Bernoulli-type Relations in Some Noncommutative Polynomial Ring

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Abstract

We find particular relations which we call "Bernoulli-type" in some noncommutative polynomial ring with a single nontrivial relation. More precisely, our ring is isomorphic to the universal enveloping algebra of a two-dimensional non-abelian Lie algebra. From these Bernoulli-type relations in our ring, we can obtain a representation on a certain left ideal with the Bernoulli numbers as structure constants.

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1 Introduction

The Bernoulli numbers are a sequence of rational numbers with connections to many branches of mathemathics. Especially, they are closely related to the values of the Riemann zeta function at negative integers [1], [2]. In this paper, we show a certain connection between some noncommutative polynomial ring and the Bernoulli numbers. We let $K[x, y]$ be a noncommutative polynomial ring in two indeterminates x, y over a field K of characteristic zero. Now, we define $I = \langle xy - yx - x \rangle$ to be the ideal of $K[x, y]$ generated by $xy - yx - x$, and let A be $K[x, y]/I$, the quotient of $K[x, y]$ by I. Again we use x, y as $\overline{x} = x + I$, $\overline{y} = y + I$ respectively (if there is no confusion). We note that A is isomorphic to the universal enveloping algebra of a two-dimensional non-abelian Lie algebra (cf. Remark 4.1). Then, our main result is the following:

Theorem (Bernoulli-type relations)
Let A be as above. We put

$$w_{k,\ell} = (xy^k - y^k x)x^\ell \in A \quad (k \geq 1, \ell \geq 0).$$

Then, the following relations hold.

$$xw_{k,\ell} = \sum_{i=1}^{k} \binom{k}{i} w_{k,\ell+1}$$
$$yw_{k,\ell} = \frac{k}{k+1} w_{k+1,\ell} - \sum_{i=1}^{k} \frac{1}{k+1} \binom{k+1}{i} B_{k+1-i} w_{i,\ell}$$

□

Here, we note that the above B_{k+1-i} mean the Bernoulli numbers. Hence, we call the above relations "Bernoulli-type relations". Put $W = \oplus_{m\geq 1, n\geq 0} Kx^m y^n \subseteq A$, which is a direct sum by PBW theorem. Then W becomes a two-sided ideal of A. Using the Bernoulli-type relations, we can obtain that W is generated by $\{w_{k,\ell}\}_{k\geq 1,\ell\geq 0}$. We can also see that $\{w_{k,\ell}\}_{k\geq 1,\ell\geq 0}$ is a basis of W.

Here we will explain Bernoulli-type relations in terms of Lie algebras. For the explanation, we start to explain our motivation of this study. We began this study with [3] written about some factorizations in universal enveloping algebras. In [3], they deal with universal enveloping algebras of three-dimensional Lie algebras. Then they obtained certain general relations. Let $\mathfrak{L}$ be a three-dimensional Lie algebra over K and denote by $U(\mathfrak{L})$ the universal enveloping algebra of $\mathfrak{L}$. Assume that $\mathfrak{L}$ is generated by two elements x, y. Then, the general relations in $U(\mathfrak{L})$ are given as follows:

$$\begin{aligned}
&(\mathrm{A}_k) \quad yxy^k \equiv \frac{k}{k+1} xy^{k+1} + \frac{1}{k+1} y^{k+1} x \quad (\mathrm{mod} U_k),\\
&(\mathrm{B}_k) \quad y^k xy \equiv \frac{1}{k+1} xy^{k+1} + \frac{k}{k+1} y^{k+1} x \quad (\mathrm{mod} U_k),\\
&(\mathrm{C}_k) \quad yU_k \subseteq U_{k+1},\ \ U_k y \subseteq U_{k+1}, \text{where}\\
&\quad U_k = \sum_{0\leq m\leq k} (Kxy^m + Ky^m x + Ky^m) \quad (k \geq 0).
\end{aligned}$$

□

Then, the remainder terms, $u = \sum_{1\leq p,q,r\leq k} a_p xy^p + b_q y^q x + c_r y^r + dx \in U_k$ with $a_p, b_q, c_r, d \in K$, of (A_k), (B_k) are determined according to the generators x, y and types of $\mathfrak{L}$. In the paper [3], they determine some exact terms of u along with a classification of $\mathfrak{L}$ in Jacobson's book [6].

Here we roughly introduce the classification. We put $\mathfrak{L} = Ke \oplus Kf \oplus Kg$ with its basis (e, f, g). Let $\mathfrak{L}'$ be the derived ideal of $\mathfrak{L}$ and $\mathfrak{C}$ be the center of $\mathfrak{L}$. Then the classification is roughly given as follows:

(a) If $\mathfrak{L}' = 0$, $\mathfrak{L}$ is abelian.

(b) If $\dim\mathfrak{L}' = 1$ and $\mathfrak{L}' \subseteq \mathfrak{C}$, the multiplication table of the basis is

$$[e, f] = g,\ \ [e, g] = [f, g] = 0.$$

(c) If $\dim\mathfrak{L}' = 1$ and $\mathfrak{L}' \not\subseteq \mathfrak{C}$, the multiplication table of the basis is

$$[e, f] = e, \ [e, g] = [f, g] = 0.$$

(d) If $\dim\mathfrak{L}' = 2$, the multiplication tables of the basis are

$$\text{(d)-}(\alpha) \quad [e, f] = 0, \ [e, g] = e, \ [f, g] = \alpha f,$$
$$\text{(d)-}(+) \quad [e, f] = 0, \ [e, g] = e + f, \ [f, g] = f,$$

where α in $K^{\times}$. Different choice of α give different algebras unless $\alpha\alpha' = 1$.

(e) $\dim\mathfrak{L}' = 3$, the multiplication table of the basis is

$$[e, f] = g, \ [g, e] = 2e, \ [g, f] = -2f.$$

In the type (d) or (e), we suppose that K is algebraically closed (just for our rough explanation). As is well-known, the type (b) means a Heisenberg Lie algebra $\mathfrak{H}_K$ and the type (e) means a special linear Lie algebra $\mathfrak{sl}_2(K)$. In the paper [3], they determined the exact terms of u for $\mathfrak{H}_F$ or $\mathfrak{sl}_2(F)$ with the above generators e, f including the case if F is a field of characteristic zero. They also showed that $\mathfrak{L}$ can not be two generated if $\mathfrak{L}$ is the type (a) or the type (d)-($\alpha = 1$). Hence, we were interested in determining the terms in U_k for the remaining type of $\mathfrak{L}$. For our purpose, we explain some results in the author's master thesis [10]. Since the paper is written in Japanese, we introduce its summary here. In [10], we obtained some properties between u and the types of $\mathfrak{L}$, and determined the exact terms of u if $\mathfrak{L}$ is the type (d)-(+). The properties between u and the types of $\mathfrak{L}$ are given as follows:

- We always have $u = 0$ inspite of generators if $\mathfrak{L}$ is the type (b).
- We always have $u \neq 0$ inspite of generators if $\mathfrak{L}$ is the type (e).
- We can get $u = 0$ according to some special generators if $\mathfrak{L}$ is the type (c) or (d). (It means that we can also get $u \neq 0$ according to another generators.)

The exact terms of u are determined if $\mathfrak{L}$ is the type (d)-(+) with the generators e and g. The formulas in $U(\mathfrak{L})$ are given as follows:

$$(\mathrm{P}_k) \quad geg^k = \frac{k}{k+1}eg^{k+1} + \frac{1}{k+1}g^{k+1}e - eg^k + \frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}g^i e,$$

$$(\mathrm{Q}_k) \quad g^k eg = \frac{1}{k+1}eg^{k+1} + \frac{k}{k+1}g^{k+1}e + \frac{1}{k+1}\sum_{i=0}^{k}(-1)^{k+1-i}\binom{k+1}{i}eg^i + g^k e.$$

□

These are the almost all results written in [10]. After we obtained these results, we could establish the formulas if $\mathfrak{L}$ is the type (c) with the generators $e + g$ and $f + g$

in the above classification. Then we noticed that our formulas can be reduced to the two-dimensional case. That is, we put $L = Kx \oplus Ky$ as a two-dimensional Lie algebra satisfying $[x, y] = x$ and denote by $U(L)$ the universal enveloping algebra of L. Then, the formulas in $U(L)$ are given as follows:

$$
\begin{aligned}
(\mathrm{P}_k)\quad yxy^k &= \frac{k}{k+1}\, xy^{k+1} + \frac{1}{k+1}\, y^{k+1}x \\
&\quad - \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_{k+1-i}\, xy^i + \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_{k+1-i}\, y^i x, \\
(\mathrm{Q}_k)\quad y^k xy &= \frac{1}{k+1}\, xy^{k+1} + \frac{k}{k+1}\, y^{k+1}x \\
&\quad + \frac{1}{k+1}\sum_{i=1}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\, xy^i \\
&\quad - \frac{1}{k+1}\sum_{i=1}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\, y^i x.
\end{aligned}
$$

□

At first these formulas were shown without the Bernoulli-type relations. But using the Bernoulli-type relations, we can easily show the formulas. Therefore, we use the Bernoulli-type relations to show the formulas in this paper.

We will review the Bernoulli numbers B_n with $B_1 = 1/2$ in Section 2. In Section 3, we will show the Bernoulli-type relations and study W introduced before. In Section 4, we will show the above formulas and explain a connection to Lie algebras. We also mention that $U(L)$ is isomorphic to A, and that $A = \oplus_{m\geq 1, n\geq 0} Kx^m yn$ by PBW theorem.

2 Preliminaries

In this paper, K is a field of characteristic zero. We denote a left hand side (resp: right hand side) by (LHS) (resp: (RHS)). We also denote by B_n the Bernoulli numbers.

In this section, we review the Bernoulli numbers with $B_1 = 1/2$. We aim a self-contained explanation in this paper. Thus we confirm our setting here.

Definition 2.1. (The Bernoulli numbers)

We define the Bernoulli numbers B_n recursively as follows:

$$\sum_{i=0}^{n}\binom{n+1}{i}B_i = n+1.$$

□

Remark 2.2. *In general, the Bernoulli numbers are also given by a generating function. The generating function in our condition is given as follows:*

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

$\square$

Here we describe the Bernoulli numbers up to $n = 10$.

Figure.1 The Bernoulli numbers

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

As well known, there are the other type of the Bernoulli numbers. If we denote by $\hat{B}_n$ the Bernoulli numbers with $\hat{B}_1 = -1/2$, then $\hat{B}_n$ are given by $(-1)^n B_n$ for $n \geq 0$.

Remark 2.3. *In the first half of eighteenth century, the Bernoulli numbers were discoverd around the same time by Jacob Bernoulli and Kowa Seki independently. At first, both Bernoulli and Seki took $B_1 = 1/2$. Hence, historically, our definition is an original version.* $\square$

3 Bernoulli-type relations and the ideal W

In this section, we show the main theorem and some corollaries. Now, we set $A = K[x, y]/I$, where $K[x, y]$ is a noncommutative polynomial ring in two indeterminates x, y and $I = \langle xy - yx - x \rangle$ is the two-sided ideal of $K[x, y]$ generated by $xy - yx - x$. At first, we confirm several elementary formulas for proving the main theorem.

Proposition 3.1. (i) *For integers $k \geq i \geq j \geq 0$, we have*

$$\binom{k}{i}\binom{i}{j} = \binom{k}{j}\binom{k-j}{i-j}.$$

(ii) *Let A be as above. Then the following formula holds.*

$$xy^k = \sum_{i=0}^{k} \binom{k}{i} y^i x$$

(iii) *Let A be as above. Then the following formula holds.*

$$y^k x = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} xy^i$$

Proof. (i) We can calculate

$$\begin{aligned}\binom{k}{i}\binom{i}{j} &= \frac{k!}{(k-i)!i!}\,\frac{i!}{(i-j)!j!}\\ &= \frac{k!}{(k-j)!j!}\,\frac{(k-j)!}{\{(k-j)-(i-j)\}!(i-j)!}\\ &= \binom{k}{j}\binom{k-j}{i-j}.\end{aligned}$$

(ii) Since $xy = yx + x = y(x+1)$, we can calculate

$$\begin{aligned}xy^k &= (yx+x)y^{k-1} = (y+1)xy^{k-1}\\ &= \dots\\ &= (y+1)^k x = \sum_{i=0}^{k}\binom{k}{i}y^i x.\end{aligned}$$

(iii) Since $yx = xy - x = x(y-1)$, we can calculate

$$\begin{aligned}y^k x &= y^{k-1}(xy - x) = y^{k-1}x(y-1)\\ &= \dots\\ &= x(y-1)^k = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}xy^i.\end{aligned}$$

Therefore, we obtain the desired results. □

Now, we prove the main theorem.

Theorem 3.2. *Let A be as above. We take*

$$w_{k,\ell} = (xy^k - y^k x)x^\ell \in A \quad (k \geq 1, \ell \geq 0).$$

Then, the following relations hold.

$$(BR1) \quad xw_{k,\ell} = \sum_{i=1}^{k}\binom{k}{i}w_{k,\ell+1}$$

$$(BR2) \quad yw_{k,\ell} = \frac{k}{k+1}w_{k+1,\ell} - \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_{k+1-i}w_{i,\ell}$$

Proof. At first, we show (BR1). Using Proposition 3.1 (ii), we can compute

$$
\begin{aligned}
xw_{k,\ell} &= x(xy^k - y^k x)x^\ell \\
&= \{x(xy^k) - (xy^k)x\}x^\ell \\
&= \left\{x\left(\sum_{i=0}^{k}\binom{k}{i}y^i x\right) - \left(\sum_{i=0}^{k}\binom{k}{i}y^i x\right)x\right\}x^\ell \\
&= \left\{\sum_{i=0}^{k}\binom{k}{i}xy^i - \sum_{i=0}^{k}\binom{k}{i}y^i x\right\}x^{\ell+1} \\
&= \sum_{i=0}^{k}\binom{k}{i}(xy^i - y^i x)x^{\ell+1} \\
&= \sum_{i=0}^{k}\binom{k}{i}w_{i,\ell+1}.
\end{aligned}
$$

Next, we show (BR2) by computing from (RHS) to (LHS). Using Proposition 3.1 (ii), we can compute

$$
\begin{aligned}
(RHS) &= \frac{k}{k+1}w_{k+1,\ell} - \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_{k+1-i}w_{i,\ell} \\
&= \frac{k}{k+1}(xy^{k+1} - y^{k+1}x)x^\ell - \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_{k+1-i}(xy^i - y^i x)x^\ell \\
&= \frac{k}{k+1}\left\{\sum_{i=0}^{k+1}\binom{k+1}{i}y^i x - y^{k+1}x\right\}x^\ell \\
&\quad - \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_{k+1-i}\left\{\sum_{j=0}^{i}\binom{i}{j}y^j x - y^i x\right\}x^\ell \\
&= \frac{k}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}y^i x^{\ell+1} - \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_{k+1-i}\sum_{j=0}^{i-1}\binom{i}{j}y^j x^{\ell+1}.
\end{aligned}
$$

We divide (RHS) into three terms such as x and $y^k x$ and otherwise. Then we have

$$
\begin{aligned}
(RHS) &= \frac{k}{k+1}\binom{k+1}{k}y^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i}y^i x^{\ell+1} + \frac{k}{k+1}\binom{k+1}{0}y^0 x^{\ell+1} \\
&\quad - \frac{1}{k+1}\sum_{i=2}^{k}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}\binom{i}{j}y^j x^{\ell+1} - \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_{k+1-i}\binom{i}{0}y^0 x^{\ell+1}.
\end{aligned}
$$

Since we can replace B_{k+1-i} with B_i in the last term, we have

$$\begin{aligned}(RHS) &= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i}y^i x^{\ell+1} + \frac{k}{k+1}x^{\ell+1} \\ &\quad - \frac{1}{k+1}\sum_{i=2}^{k}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}\binom{i}{j}y^j x^{\ell+1} - \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_i x^{\ell+1} \\ &= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i}y^i x^{\ell+1} - \frac{1}{k+1}\sum_{i=2}^{k}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}\binom{i}{j}y^j x^{\ell+1} \\ &\quad + \frac{k}{k+1}x^{\ell+1} - \frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}B_i x^{\ell+1} + \frac{1}{k+1}\binom{k+1}{0}x^{\ell+1}.\end{aligned}$$

In the fifth term, using Definition 2.1, we get

$$\begin{aligned}(RHS) &= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i}y^i x^{\ell+1} - \frac{1}{k+1}\sum_{i=2}^{k}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}\binom{i}{j}y^j x^{\ell+1} \\ &\quad + \frac{k}{k+1}x^{\ell+1} - \frac{1}{k+1}(k+1)x^{\ell+1} + \frac{1}{k+1}x^{\ell+1}. \\ &= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i}y^i x^{\ell+1} - \frac{1}{k+1}\sum_{i=2}^{k}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}\binom{i}{j}y^j x^{\ell+1}.\end{aligned}$$

Replacing the index i with $i+1$ in the third term, we obtain

$$(RHS) = ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i}y^i x^{\ell+1} - \frac{1}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i+1}B_{k-i}\sum_{j=1}^{i}\binom{i+1}{j}y^j x^{\ell+1}.$$

Then, changing additive method in the third term, we obtain

$$(RHS) = ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i}y^i x^{\ell+1} - \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=j}^{k-1}\binom{k+1}{i+1}\binom{i+1}{j}B_{k-i}\right\}y^j x^{\ell+1}.$$

In the third term, using Proposition 3.1 (i), we get

$$\begin{aligned}(RHS) &= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i}y^i x^{\ell+1} \\ &\quad - \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=j}^{k-1}\binom{k+1}{j}\binom{k+1-j}{i+1-j}B_{k-i}\right\}y^j x^{\ell+1}.\end{aligned}$$

Then, replacing the index $i+1-j$ with i , we get

$$
\begin{aligned}
(RHS) &= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1} \\
&\quad - \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=1}^{k-j}\binom{k+1}{j}\binom{k+1-j}{i} B_{k-(i+j-1)}\right\} y^j x^{\ell+1} \\
&= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1} \\
&\quad - \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=1}^{k-j}\binom{k+1}{j}\binom{k-j+1}{i} B_{k-j+1-i}\right\} y^j x^{\ell+1}.
\end{aligned}
$$

Since we have $\binom{k-j+1}{i} = \binom{k-j+1}{k-j+1-i}$, we can replace $B_{k-j+1-i}$ with B_i. Hence we have

$$
\begin{aligned}
(RHS) &= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1} \\
&\quad - \frac{1}{k+1}\sum_{j=1}^{k-1}\binom{k+1}{j}\left\{\sum_{i=1}^{k-j}\binom{k-j+1}{i} B_i\right\} y^j x^{\ell+1} \\
&= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1} \\
&\quad - \frac{1}{k+1}\sum_{j=1}^{k-1}\binom{k+1}{j}\left\{\sum_{i=0}^{k-j}\binom{k-j+1}{i} B_i - \binom{k+j-1}{0} B_0\right\} y^j x^{\ell+1}.
\end{aligned}
$$

In the third term, using Definition 2.1, we get

$$
(RHS) = ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i} y^i x^{\ell+1} - \frac{1}{k+1}\sum_{j=1}^{k-1}\binom{k+1}{j}\{(k-j+1)-1\} y^j x^{\ell+1}.
$$

Then, replacing the index j with i in the third term, we have

$$
\begin{aligned}
(RHS) &= ky^k x^{\ell+1} + \frac{k}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i}y^i x^{\ell+1} - \frac{1}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i}(k-i)y^i x^{\ell+1}\\
&= ky^k x^{\ell+1} + \frac{1}{k+1}\sum_{i=1}^{k-1}k\binom{k+1}{i}y^i x^{\ell+1} - \frac{1}{k+1}\sum_{i=1}^{k-1}\binom{k+1}{i}(k-i)y^i x^{\ell+1}\\
&= ky^k x^{\ell+1} + \frac{1}{k+1}\sum_{i=1}^{k-1}i\binom{k+1}{i}y^i x^{\ell+1}\\
&= \binom{k}{k-1}y^k x^{\ell+1} + \sum_{i=1}^{k-1}\binom{k}{i-1}y^i x^{\ell+1}\\
&= \sum_{i=1}^{k}\binom{k}{i-1}y^i x^{\ell+1}.
\end{aligned}
$$

Replacing the index i with $i+1$, we get

$$
= \sum_{i=0}^{k-1}\binom{k}{i}y^{i+1}x^{\ell+1}.
$$

Regarding $y^{i+1}x^{\ell+1}$ as $y(y^i x^{\ell+1})$, we have

$$
\begin{aligned}
&= y\sum_{i=0}^{k-1}\binom{k}{i}y^i x^{\ell+1}\\
&= y\left(\sum_{i=0}^{k}\binom{k}{i}y^i x^{\ell+1} - y^k x^{\ell+1}\right)\\
&= y\left(\sum_{i=0}^{k}\binom{k}{i}y^i x - y^i x\right)x^{\ell}\\
&= y(xy^k - y^k x)x^{\ell}\\
&= yw_{k,\ell} = (LHS).
\end{aligned}
$$

Therefore, we obtain desired results. □

From the theorem, we can get some corollaries. As has been mentioned in the introduction, A is isomorphic to the universal enveloping algebra of a two-dimensional non-abelian Lie algebra. Thus, using PBW theorem, we can put

$$
W = \bigoplus_{m\geq 1, n\geq 0} Kx^m y^n.
$$

Here we put

$$W' = \left\{ \sum_{k,\ell} c_{k,\ell} w_{k,\ell} \;\middle|\; \begin{array}{c} k \geq 1, \ell \geq 0, c_{k,\ell} \in K, \\ c_{k,\ell} = 0 \text{ for all but finitely many pairs } (k,\ell) \end{array} \right\}.$$

Then, the following statements hold.

Corollary 3.3. *Notation is as above. Then, W' is a two-sided ideal of A. In particular, $W = W'$.*

Proof. From Theorem 3.2, it is clear that W' becomes a left ideal of A. Again using Theorem 3.2, we can see

$$W' = Aw_{1,0} = Ax.$$

Then, we have

$$W'x = (Ax)x \subseteq W'$$

and

$$W'y = (Ax)y = A(xy) = A(yx + x) = A(y + 1)x \subseteq W'.$$

Hence, W' is a two-sided ideal of A. Using Proposition 3.1, we can obtain

$$x^m y^n = x^{m-1}(xy^n) = x^{m-1}\left(\sum_{i=0}^{n} \binom{n}{i} y^i\right)x,$$

which implies $W = Ax$ and $W = W'$. Therefore, we obtain the desired result. □

Next, we see that $\{w_{k,\ell}\}_{k\geq 1,\ell\geq 0}$ is a basis of W'.

Corollary 3.4. *Notation is as above. Then, $\{w_{k,\ell}\}_{k\geq 1,\ell\geq 0}$ is a basis of W, that is, $W = \oplus_{k\geq 1,\ell\geq 0} K w_{k,\ell}$.*

Proof. We show $\{w_{k,\ell}\}_{k\geq 1,\ell\geq 0}$ to be linearly independent. We assume

$$\sum_{\ell=1}^{n} \sum_{k=1}^{m} c_{k,\ell}(xy^k - y^k x)x^\ell = 0 \quad (m, n < \infty)$$

with $c_{k,\ell} \in K$. Then, from Proposition 3.1 (ii), we obtain

$$\begin{aligned} (LHS) &= \sum_{\ell=1}^{n} \sum_{k=1}^{m} c_{k,\ell}(xy^k - y^k x)x^\ell \\ &= \sum_{\ell=1}^{n} \sum_{k=1}^{m} c_{k,\ell} \sum_{i=0}^{k-1} \binom{k}{i} y^i x^{\ell+1}. \end{aligned}$$

Hence, we have

$$
\begin{aligned}
(LHS) &= \sum_{\ell=1}^{n} c_{m,\ell} \sum_{i=0}^{m-1} \binom{m}{i} y^i x^{\ell+1} + \sum_{\ell=1}^{n} \sum_{k=1}^{m-1} c_{k,\ell} \sum_{i=0}^{k-1} \binom{k}{i} y^i x^{\ell+1} \\
&= \sum_{\ell=1}^{n} c_{m,\ell} \binom{m}{m-1} y^{m-1} x^{\ell+1} + \sum_{\ell=1}^{n} c_{m,\ell} \sum_{i=0}^{m-2} \binom{m}{i} y^i x^{\ell+1} + \sum_{\ell=1}^{n} \sum_{k=1}^{m-1} c_{k,\ell} \sum_{i=0}^{k-1} \binom{k}{i} y^i x^{\ell+1}.
\end{aligned}
$$

Then, the term $y^{m-1}x^{\ell+1}$ is appeared in the first term only. Using PBW theorem, we can get $c_{m,\ell} = 0$ for all ℓ. Hence, the second term is vanished. That is, we have

$$
(LHS) = \sum_{\ell=1}^{n} \sum_{k=1}^{m-1} c_{k,\ell} \sum_{i=0}^{k-1} \binom{k}{i} y^i x^{\ell+1}.
$$

Continuing this operation, we get $c_{k,\ell} = 0$ for all k. Namely, we get $c_{k,\ell} = 0$ for all k and ℓ. Hence, $\{w_{k,\ell}\}_{k\geq 1,\ell\geq 0}$ is a basis of W. □

Next, we show a variation of the Bernoulli-type relations.

Corollary 3.5. *Let A be as above. We take*

$$
w_k = xy^k - y^k x \in A \quad (k \geq 1).
$$

Then, the following relations hold.

$$
\begin{aligned}
&(SBR1) \quad yw_k = \frac{k}{k+1} w_{k+1} - \frac{1}{k+1} \sum_{i=1}^{k} \binom{k+1}{i} B_{k+1-i} w_i \\
&(SBR2) \quad w_k y = \frac{k}{k+1} w_{k+1} - \frac{1}{k+1} \sum_{i=1}^{k} (-1)^{k+1-i} \binom{k+1}{i} B_{k+1-i} w_i
\end{aligned}
$$

Proof. In Theorem 3.2, if we take $\ell = 0$, then (SBR1) holds.

Next, we show (BR2) by computing from (RHS) to (LHS). Using Proposition 3.1

(iii), we can compute

$$
\begin{aligned}
(RHS) &= \frac{k}{k+1}w_{k+1} - \frac{1}{k+1}\sum_{i=1}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}w_i \\
&= \frac{k}{k+1}(xy^{k+1} - y^{k+1}x) - \frac{1}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_{k+1-i}(xy^i - y^i x). \\
&= \frac{k}{k+1}\left(xy^{k+1} - \sum_{i=0}^{k+1}(-1)^{k+1-i}\binom{k+1}{i}xy^i\right) \\
&\quad - \frac{1}{k+1}\sum_{i=1}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\left(xy^i - \sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}xy^j\right) \\
&= \frac{k}{k+1}\sum_{i=0}^{k}(-1)^{k-i}\binom{k+1}{i}xy^i \\
&\quad - \frac{1}{k+1}\sum_{i=1}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\sum_{j=0}^{i-1}(-1)^{i-1-j}\binom{i}{j}xy^j.
\end{aligned}
$$

We divide (RHS) into three terms such as x and $y^k x$ and otherwise. Then we have

$$
\begin{aligned}
(RHS) &= \frac{(-1)^{k-k}k}{k+1}\binom{k+1}{k}xy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i + \frac{(-1)^{k-0}}{k+1}\binom{k+1}{0}xy^0 \\
&\quad - \frac{1}{k+1}\sum_{i=2}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}(-1)^{i-1-j}\binom{i}{j}xy^j \\
&\quad - \frac{1}{k+1}\sum_{i=1}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}(-1)^{i-1-0}\binom{i}{0}xy^0.
\end{aligned}
$$

Since we can replace B_{k+1-i} with B_i in the last term, we have

$$\begin{aligned}
(RHS) &= kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i + \frac{(-1)^k}{k+1}x \\
&\quad - \frac{1}{k+1}\sum_{i=2}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}(-1)^{i-1-j}\binom{i}{j}xy^j - \frac{(-1)^k}{k+1}\sum_{i=1}^{k}\binom{k+1}{i}B_i x \\
&= kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\
&\quad - \frac{1}{k+1}\sum_{i=2}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}(-1)^{i-1-j}\binom{i}{j}xy^j \\
&\quad + \frac{(-1)^k}{k+1}x - \frac{(-1)^k}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}B_i x - \frac{(-1)^k}{k+1}\binom{k+1}{0}B_0 x.
\end{aligned}$$

In the fifth term, using Definition 2.1, we get

$$\begin{aligned}
(RHS) &= kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\
&\quad - \frac{1}{k+1}\sum_{i=2}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}(-1)^{i-1-j}\binom{i}{j}xy^j \\
&\quad + \frac{(-1)^k}{k+1}x - \frac{(-1)^k}{k+1}(k+1)x - \frac{(-1)^k}{k+1}x \\
&= kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\
&\quad - \frac{1}{k+1}\sum_{i=2}^{k}(-1)^{k+1-i}\binom{k+1}{i}B_{k+1-i}\sum_{j=1}^{i-1}(-1)^{i-1-j}\binom{i}{j}xy^j.
\end{aligned}$$

Replacing the index i with $i+1$ in the third term, we obtain

$$\begin{aligned}
(RHS) &= kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\
&\quad - \frac{1}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i+1}B_{k-i}\sum_{j=1}^{i}(-1)^{i-j}\binom{i+1}{j}xy^j.
\end{aligned}$$

Then, changing additive method in the third term, we obtain

$$\begin{aligned}(RHS) = kxy^k &+ \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ &- \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=j}^{k-1}(-1)^{k-j}\binom{k+1}{i+1}\binom{i+1}{j}B_{k-i}\right\}xy^j.\end{aligned}$$

In the third term, using Proposition 3.1 (i), we get

$$\begin{aligned}(RHS) = kxy^k &+ \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ &- \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=j}^{k-1}(-1)^{k-j}\binom{k+1}{j}\binom{k+1-j}{i+1-j}B_{k-i}\right\}xy^j.\end{aligned}$$

Then, replacing the index $i+1-j$ with i , we get

$$\begin{aligned}(RHS) = kxy^k &+ \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ &- \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=1}^{k-j}(-1)^{k-j}\binom{k+1}{j}\binom{k+1-j}{i}B_{k-(i+j-1)}\right\}xy^j \\ = kxy^k &+ \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ &- \frac{1}{k+1}\sum_{j=1}^{k-1}\left\{\sum_{i=1}^{k-j}(-1)^{k-j}\binom{k+1}{j}\binom{k-j+1}{i}B_{k-j+1-i}\right\}xy^j.\end{aligned}$$

Since we have $\binom{k-j+1}{i} = \binom{k-j+1}{k-j+1-i}$, we can replace $B_{k-j+1-i}$ with B_i. Hence we have

$$\begin{aligned}(RHS) = kxy^k &+ \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ &- \frac{1}{k+1}\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k+1}{j}\left\{\sum_{i=1}^{k-j}\binom{k-j+1}{i}B_i\right\}xy^j \\ = kxy^k &+ \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i \\ &- \frac{1}{k+1}\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k+1}{j}\left\{\sum_{i=0}^{k-j}\binom{k-j+1}{i}B_i - \binom{k-j+1}{0}B_0\right\}xy^j.\end{aligned}$$

In the third term, using Definition 2.1, we get

$$(RHS) = kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i - \frac{1}{k+1}\sum_{j=1}^{k-1}(-1)^{k-j}\binom{k+1}{j}\{(k-j+1)-1\}xy^j.$$

Then, replacing the index j with i in the third term, we have

$$\begin{aligned}
(RHS) &= kxy^k + \frac{k}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}xy^i - \frac{1}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}\binom{k+1}{i}(k-i)xy^j \\
&= kxy^k + \frac{1}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}k\binom{k+1}{i}xy^i - \frac{1}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}(k-i)\binom{k+1}{i}xy^j \\
&= kxy^k + \frac{1}{k+1}\sum_{i=1}^{k-1}(-1)^{k-i}i\binom{k+1}{i}xy^i \\
&= \binom{k}{k-1}xy^k + \sum_{i=1}^{k-1}(-1)^{k-i}\binom{k}{i-1}xy^i \\
&= \sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i-1}xy^i.
\end{aligned}$$

Replacing the index i with $i+1$, we get

$$(RHS) = \sum_{i=0}^{k-1}(-1)^{k+1-i}\binom{k}{i}xy^{i+1}.$$

Regarding $y^{i+1}x^{\ell+1}$ as $y(y^i x^{\ell+1}$, we have

$$\begin{aligned}
(RHS) &= \left(\sum_{i=0}^{k-1}(-1)^{k+1-i}\binom{k}{i}xy^i\right)y \\
&= \left(xy^k - \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}xy^i\right)y \\
&= (xy^k - y^k x)y \\
&= w_k y = (LHS).
\end{aligned}$$

Therefore, we obtain desired results. □

We can easily see that w_k is $w_{k,0}$ in Theorem 3.2. We will investigate connections between Bernoulli-type relations and Lie algebras in the next section. Using Corollary 3.5, we will show the formulas with respect to Lie algebras.

4 A Connection between Bernoulli-type relations and Lie algebras

In this section, we consider a connection between the Bernoulli-type relations and Lie algebras. In the introduction, we roughly reviewed the classification of three-dimensional Lie algebras. We let $\mathfrak{L}$ be a three-dimensional Lie algebra over a field K of characteristic zero and denote by $U(\mathfrak{L})$ the universal enveloping algebra of $\mathfrak{L}$. Then we also explain that if $\mathfrak{L}$ is the type (c), we have a two-dimensional Lie subalgebra L of $\mathfrak{L}$. Then, L is a non-abelian two-dimensional Lie algebra. That is, we can write $L = Kx \oplus Ky$ with $[x, y] = x$.

Now, we recall our settings in Section 3. We let $K[x, y]$ be a noncommutative polynomial ring generated by x, y and define $I = \langle xy - yx - x \rangle$ to be the ideal of $K[x, y]$ generated by $xy - yx - x$. We let A be $K[x, y]/I$. Then, if we denote by $U(L)$ the universal enveloping algebra of L, then we can see the following :

Remark 4.1. *Notation is as above. Then we have* $A \cong U(L)$*.* □

From Remark 4.1, we can use the Bernoulli-type relations for $U(L)$. Conversely, it is the reason that we can use PBW theorem in A. Using the relations in Section 3, we will show the next formulas in $U(L)$.

Proposition 4.2. *Let L be as above. Then in* $U(L)$*, we have*

$$\begin{aligned}
(\mathrm{P}_k)\quad yxy^k &= \frac{k}{k+1}\, xy^{k+1} + \frac{1}{k+1}\, x^{k+1} x \\
&\quad - \frac{1}{k+1} \sum_{i=1}^{k} \binom{k+1}{i} B_{k+1-i}\, xy^i + \frac{1}{k+1} \sum_{i=1}^{k} \binom{k+1}{i} B_{k+1-i}\, y^i x, \\
(\mathrm{Q}_k)\quad y^k xy &= \frac{1}{k+1}\, xy^{k+1} + \frac{k}{k+1}\, y^{k+1} x \\
&\quad + \frac{1}{k+1} \sum_{i=1}^{k} (-1)^{k+1-i} \binom{k+1}{i} B_{k+1-i}\, xy^i \\
&\quad - \frac{1}{k+1} \sum_{i=1}^{k} (-1)^{k+1-i} \binom{k+1}{i} B_{k+1-i}\, y^i x.
\end{aligned}$$

Proof. Using the Corollary 3.5, we see that (SBR1) implies (P_k) and (SBR2) implies (Q_k). □

Remark 4.3. *The above formulas* (P_k) *and* (Q_k) *completely give the remaining terms of* (A_k) *and* (B_k) *in case of the type (c) if we replace* x, y *by* $e + g$, $f + g$ *respectively.*

Remark 4.4. *Using the theory of linear algebras, we can establish two-dimensional Lie algebras as follows:*

Let V *be a vector space over* K*, and* $End(V)$ *be its endmorphism ring. Put* $\mathfrak{g} = End(V) \oplus V$*, and we define*

$$[\, f_1 + v_1, f_2 + v_2] = (f_1 f_2 - f_2 f_1) + (f_1(v_2) - f_2(v_1))$$

for all $f_1, f_2 \in End(V)$ *and* $v_1, v_2 \in V$. *Then* $\mathfrak{g}$ *becomes a Lie algebra. Suppose that* $f \in End(V)$ *and* $v \in V$ *satisfy* $f(v) = cv$ *for some* $c \in K$. *Put* $\mathfrak{a} = Kf \oplus Kv$ *as a Lie subalgebra of* $\mathfrak{g}$. *Then, we have*

$$\begin{cases} \mathfrak{a} \text{ is abelian} & (\text{if } c = 0), \\ \mathfrak{a} \cong L & (\text{if } c \neq 0). \end{cases}$$

Remark 4.5. *If K is algebraically closed, then three-dimensional Lie algebras of type (d) corresponding to* $\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$ *in Jacobson's book [6], on page 12, are not according to* β. *Hence, in this paper, we introduce the exact one type as (d)-(+) at the introduction.*

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